CBMeMBer filters for nonstandard targets, I: Extended targets

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***Abstract*—The cardinality balanced multi-target multi-Bernoulli (CBMeMBer) recursion approximates the multi-target posterior density by a multi-Bernoulli probability density, and is based on the standard measurement model that targets are points. In reality, there are many other measurement models substantially different from this. This paper represents part of our theoretical studies on a CBMeMBer filter for nonstandard multi-target measurement models. The main contribution of our work lies in the establishment of the measurement-update equations for a CBMeMBer filter based on the Poisson extended-target measurement model proposed by Gilholm et al.**

Keywords—Multi-target tracking, extended targets, random finite set (RFS), CBMeMBer filter.

I. INTRODUCTION

In a multi-target tracking scenario, both the number and states of targets need to be estimated from a sequence of observation sets in the presence of detection uncertainty, data association uncertainty and clutter. Mahler's Bayesian approach, based on the finite set statistics (FISST) theory [1], [2], has generated substantial research interest, because we can obtain both the number and state estimates of targets by this method without data association required in conventional multi-target tracking methods. However, the Bayesian approach is usually analytically intractable due to the inherent combinatorial nature of multi-target densities and the multiple integrations on the (infinite dimensional) multi-target state and observation spaces.

To address this issue, some random finite set (RFS) filters have been developed. For instance, the probability hypothesis density (PHD) filter [3] and the cardinality PHD (CPHD) filter [4], proposed by Mahler, propagate the first-order statistical moment as well as the cardinality distribution of the posterior density. Sequential Monte Carlo (SMC) and Gaussian mixture (GM) implementations for both the PHD and CPHD filters [5], [6], [7], [8] make it possible for practical applications [9], [10], [11], [12], [13], [14]. In addition to the PHD and CPHD filters, Mahler put forward the multi-target multi-Bernoulli (MeMBer) filter in [2], which approximates the multi-target posterior density by a multi-Bernoulli probability density. But the MeMBer filter has a significant bias in the number of targets. Then Vo et al. proposed a cardinality balanced MeMBer (CBMeMBer) filter [15] to reduce the cardinality bias. Compared with the PHD and CPHD filters, the key advantage of the CBMeMBer filter is that its SMC implementation does not require clustering to extract state estimates from the particle population, which is expensive and unreliable. Thus the CBMeMBer filter is a promising method to deeply study.

However, there is a significant limitation for the above filters that they are based on the standard multi-target measurement model, in which targets are points. In reality, there are many other measurement models substantially different from this. Mahler summarized several models in [2] as follows:

- Poisson extended-target model (pp. 431-432) proposed by Gilholm et al. [16];
- unresolved-target model with continuous target number (pp. 432-444);
- bearing-only sensor measurement model (pp. 448-452) proposed by Vihola [17];
- a model for data-cluster extraction (pp. 452-458) proposed by Cheeseman [18];
- a model for multiuser detection (MUD) in communications networks (p. 550) proposed by Biglieri and Lops [19].

Are the RFS filters still useful for these nonstandard models? Of course, we hope the FISST can also analyze these cases. Mahler has made a lot of theoretical studies about the PHD and CPHD filters for the nonstandard models [20], [21], [22], [23], [24], proving the effectiveness of the FISST for these cases. This paper is one part of our theoretical studies on a CBMeMBer filer for nonstandard multi-target measurement models.

In terms of the ranges between targets and a sensor as well as the sensor resolution, there are three different measurement models in practical applications [2],

1) When targets are sufficiently close to a sensor, they may have detectable physical extent. That is, each target will generate more than one measurement. Such targets are called *extended targets*.
2) When further away, they diminish sufficiently in extent that they can be idealized as a group of mathematical points. Such targets are called *point targets*.
3) When at even greater distance, multiple point targets are diminished even further and can be further idealized as a point cluster. Such targets are called *unresolved targets*.

Mahler gave the theoretical results of a PHD filter for extended targets and unresolved targets in [20] and [21], respectively. Then Granström et al. illustrated the PHD filter for extended targets by GM implementation in [25] and [26]. Moreover, Orguner et al. presented a CPHD filter for extended-target tracking in [27].

In this paper, we generalize the CBMeMBer filter to the Poisson extended-target measurement model proposed by Gilholm et al. [16]. The main contribution of our work lies in the establishment of the measurement-update equations for a CBMeMBer filter based on the Poisson measurement model. Although these equations are theoretically rigorous and exact, they are combinatorial in form and computationally intractable as they stand. We will focus on the tractable approaches in the future research.

The remaining part of this paper is organized as follows. The CBMeMBer filter is briefly reviewed in Section II. The measurement-update equations for a CBMeMBer recursion based on the Poisson extended-target measurement model are given in Section III. The corresponding theoretical derivations of the measurement-update equations, using the FISST theory, are given in Section IV. Finally, Section V gives concluding remarks and outlooks in the future work.

II. The CBMeMBer recursion for the point-target model (review)

The CBMeMBer recursion approximates the multi-target posterior density by a multi-Bernoulli probability density with parameters $\pi = \left\{ \left(r^{(i)}, p^{(i)} \right) \right\}_{i=1}^{M}$, where $r^{(i)}$ denotes the existing probability of a single target, $p^{(i)}$ denotes the space probability density, and the mean cardinality is $\sum_{i=1}^{M} r^{(i)}$. Moreover, this recursion needs to meet the following conditions:

- Each target evolves and generates observations independently;
- Birth targets follow a multi-Bernoulli RFS independent of surviving targets;
- Clutter with a low density follows a Poisson RFS independent of target-originated observations.

The CBMeMBer recursion contains the following two parts.

A. Prediction

Assume that the multi-target posterior density is a multi-Bernoulli with parameters $\pi_{k-1} = \left\{ \left(r_{k-1}^{(i)}, p_{k-1}^{(i)} \right) \right\}_{i=1}^{M_{k-1}}$ at time $k-1$, then the predicted density is still a multi-Bernoulli and can be obtained by

$$\pi_{k|k-1} = \left\{ \left(r_{P,k|k-1}^{(i)}, p_{P,k|k-1}^{(i)} \right) \right\}_{i=1}^{M_{k-1}} \cup \left\{ \left(r_{\Gamma,k}^{(i)}, p_{\Gamma,k}^{(i)} \right) \right\}_{i=1}^{M_{\Gamma,k}} \tag{1}$$

where

$$r_{P,k|k-1}^{(i)} = r_{k-1}^{(i)} p_{k-1}^{(i)} \left[p_{S,k} \right] \tag{2}$$

$$p_{P,k|k-1}^{(i)}(x) = \frac{p_{k-1}^{(i)} \left[p_{S,k} f_{k|k-1}(x|\cdot) \right]}{p_{k-1}^{(i)} \left[p_{S,k} \right]} \tag{3}$$

and where $p_{S,k}(\cdot)$ is the survival probability, $f_{k|k-1}(\cdot|\cdot)$ is the Markov transition density for a single target, $\left\{ \left(r_{\Gamma,k}^{(i)}, p_{\Gamma,k}^{(i)} \right) \right\}_{i=1}^{M_{\Gamma,k}}$ are parameters of a multi-Bernoulli for birth targets at time k, $p[h] = \int h(x) p(x) dx$.

In essence, the multi-Bernoulli parameter set for the predicted multi-target density $\pi_{k|k-1}$ is a union of the multi-Bernoulli parameter sets for the surviving targets (the first term in Eq. (1)) and birth targets (the second term in Eq. (1)). The total number of predicted hypothesized tracks is $M_{k|k-1} = M_{k-1} + M_{\Gamma,k}$.

B. Update

Assume that the multi-target predicted density is a multi-Bernoulli with parameters $\pi_{k|k-1} = \left\{ \left(r_{k|k-1}^{(i)}, p_{k|k-1}^{(i)} \right) \right\}_{i=1}^{M_{k|k-1}}$, then the posterior density can be approximated by

$$\pi_k \approx \left\{ \left(r_{L,k}^{(i)}, p_{L,k}^{(i)} \right) \right\}_{i=1}^{M_{k|k-1}} \cup \left\{ \left(r_{U,k}(z), p_{U,k}(\cdot; z) \right) \right\}_{z \in Z_k} \tag{4}$$

where

$$r_{L,k}^{(i)} = \frac{r_{k|k-1}^{(i)} \left(1 - p_{k|k-1}^{(i)} \left[p_{D,k} \right] \right)}{1 - r_{k|k-1}^{(i)} p_{k|k-1}^{(i)} \left[p_{D,k} \right]} \tag{5}$$

$$p_{L,k}^{(i)}(x) = \frac{1 - p_{D,k}(x)}{1 - p_{k|k-1}^{(i)} \left[p_{D,k} \right]} p_{k|k-1}^{(i)}(x) \tag{6}$$

$$r_{U,k}(z) = \frac{\sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} \left(1 - r_{k|k-1}^{(i)} \right) p_{k|k-1}^{(i)} \left[P_{D,k} \phi_z \right]}{\left(1 - r_{k|k-1}^{(i)} p_{k|k-1}^{(i)} \left[P_{D,k} \right] \right)^2}}{\lambda_k c_k(z) + \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)} \left[P_{D,k} \phi_z \right]}{1 - r_{k|k-1}^{(i)} p_{k|k-1}^{(i)} \left[P_{D,k} \right]}} \tag{7}$$

$$p_{U,k}(x; z) = \frac{\sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)}}{1 - r_{k|k-1}^{(i)}} P_{D,k}(x) \phi_z(x) p_{k|k-1}^{(i)}(x)}{\sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)}}{1 - r_{k|k-1}^{(i)}} p_{k|k-1}^{(i)} \left[P_{D,k} \phi_z \right]} \tag{8}$$

and where Z_k is the measurement set, $p_{D,k}(x)$ is the detection probability, $\phi_z(x) = f_k(z|x)$ is the single target likelihood, λ_k is the expected value of Poisson clutter and $c_k(\cdot)$ is its space probability density.

In essence, the multi-Bernoulli parameter set for the updated multi-target density π_k is a union of the multi-Bernoulli parameter sets for the legacy tracks (the first term in Eq. (4)) and measurement-corrected tracks (the second term in Eq. (4)). The total number of posterior hypothesis tracks is $M_k = M_{k|k-1} + |Z_k|$.

III. The CBMeMBer update for the Poisson extended-target model

In this section, we present the measurement-update equations for a CBMeMBer recursion based on the Poisson extended-target model proposed by Gilholm et al. The detailed theoretical derivations can be seen in Section IV.

In a multiple extended-target tracking scenario, we denote a single extended target as a Bernoulli RFS $X^{(i)}$ with parameters $\left(r^{(i)}, p^{(i)} \right)$. Thus a multi-Bernoulli RFS can be marked as $X =$

$\left\{X^{(1)}, ..., X^{(M)}\right\}$, and its probability density (pp. 368 of [2]) is

$$\pi\left(\{x_1, ..., x_n\}\right) = \prod_{j=1}^{M}\left(1 - r^{(j)}\right) \cdot \sum_{1 \le i_1 \ne ... \ne i_n \le M} \prod_{j=1}^{n} \frac{r_k^{(i_j)} p_k^{(i_j)}(x_j)}{1 - r^{(i_j)}} \tag{9}$$

The mean cardinality of X is $\sum_{i=1}^{M} r^{(i)}$.

Z denotes measurement set, and $z \in Z$ denotes a single measurement vector. The likelihood function for a single extended target [16] is

$$L_Z(x) = e^{-\gamma(x)} \prod_{z \in Z} \gamma(x)\phi_z(x) \tag{10}$$

where $\gamma(\cdot)$ is the expected number of measurements generated by a single extended target, $\phi_z(x) = f_k(z|x)$ is the likelihood function. Moreover, the CBMeMBer recursion for extended targets also needs to meet the conditions described in Section II. Then we establish the following result:

THEOREM 1. *Assume that the predicted density of extended targets is a multi-Bernoulli with parameters* $\pi_{k|k-1} = \left\{\left(r_{k|k-1}^{(i)}, p_{k|k-1}^{(i)}\right)\right\}_{i=1}^{M_{k|k-1}}$*, then the posterior density can be approximated by*

$$\pi_k \approx \left\{\left(r_{L,k}^{(i)}, p_{L,k}^{(i)}\right)\right\}_{i=1}^{M_{k|k-1}} \cup \left\{\bigcup_{\wp \angle Z_k} \left\{\left(r_{U,k}(W), p_{U,k}(\cdot; W)\right)\right\}_{W \in \wp}\right\} \tag{11}$$

where

$$r_{L,k}^{(i)} = \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]}{1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]} \tag{12}$$

$$p_{L,k}^{(i)}(x) = p_{k|k-1}^{(i)}(x) \frac{1 - \bar{p}_{D,k}(x)}{p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]} \tag{13}$$

$$r_{U,k}(W) = \frac{w_\wp}{d_W} \cdot \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)}\left(1 - r_{k|k-1}^{(i)}\right) p_{k|k-1}^{(i)}[\psi_W]}{\left(1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]\right)^2} \tag{14}$$

$$p_{U,k}(x; W) = \frac{\sum_{i=1}^{M_{k|k-1}} \upsilon_{U,k}^{(i)}(x; W)}{\sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)}\left(1 - r_{k|k-1}^{(i)}\right) p_{k|k-1}^{(i)}[\psi_W]}{\left(1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]\right)^2}} \tag{15}$$

$$\bar{p}_{D,k}(x) = p_{D,k}(x)\left(1 - e^{-\gamma(x)}\right) \tag{16}$$

$$w_\wp = \frac{\prod_{W \in \wp} d_W}{\sum_{\wp' \angle Z_k} \prod_{W \in \wp'} d_W} \tag{17}$$

$$d_W = \delta_{1,|W|} + \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}[\psi_W]}{1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]} \tag{18}$$

$$\psi_W(x) = p_{D,k}(x) e^{-\gamma(x)} \prod\nolimits_{z \in W} \frac{\gamma(x)\phi_z(x)}{\lambda c(z)} \tag{19}$$

$$\begin{aligned} &\upsilon_{U,k}^{(i)}(x; W) = \\ &p_{k|k-1}^{(i)}(x)\left(1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]\right)^{-2} \cdot \\ &\left\{\left(1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[1 - \bar{p}_{D,k}\right]\right) r_{k|k-1}^{(i)} \psi_W(x)\right. \\ &\quad \left. - \left(r_{k|k-1}^{(i)}\right)^2 p_{k|k-1}^{(i)}[\psi_W]\left(1 - \bar{p}_{D,k}(x)\right)\right\} \end{aligned} \tag{20}$$

and where $\delta_{\alpha,\beta}$ *is the Kronecker delta function (i.e., if* $\alpha = \beta$*,* $\delta_{\alpha,\beta} = 1$*; otherwise,* $\delta_{\alpha,\beta} = 0$*),* $\wp$ *is one partition of* Z_k*,* W *is one cell of* $\wp$*. Notice that* W *does not include null set.*

For instance, $Z_k = \{z_1, z_2, z_3\}$ can be partitioned as follows [20],

$$\begin{aligned} &\wp_1 = \{\{z_1\}, \{z_2\}, \{z_3\}\}, \quad \wp_2 = \{\{z_1, z_2\}, \{z_3\}\}, \\ &\wp_3 = \{\{z_1\}, \{z_2, z_3\}\}, \qquad \wp_4 = \{\{z_1, z_3\}, \{z_2\}\}, \\ &\wp_5 = \{\{z_1, z_2, z_3\}\}. \end{aligned} \tag{21}$$

Besides, $e^{-\gamma(x)}$ is the probability that a target with state x will generate no detections. $1 - e^{-\gamma(x)}$ is the probability that it will generate at least one detection. Thus, taking $p_{D,k}(x)$ into account, $\bar{p}_{D,k}(x) = p_{D,k}(x)\left(1 - e^{-\gamma(x)}\right)$ is the effective probability of detection. The total number of posterior hypothesis tracks is $M_k = M_{k|k-1} + \sum_{\wp \angle Z_k} |\wp|$.

IV. THEORETICAL DERIVATIONS

In this section we present the proof of Theorem 1, beginning with a "road map" of the proof in Section IV.A. The proof of Theorem 1 is presented in Section IV.D. This proof is on the basis of Lemma 2, which is proved in Section IV.C. Moreover, Lemma 2 is on the basis of Lemma 1, which is proved in Section IV.B.

A. "Road map" of the proof

The probability generating functional (PGFL) of the multi-target Bayesian update formula (Eq. (14.280), pp. 530 of [2]) is

$$G_k[h] = \frac{\delta F}{\delta Z_k}[0, h] \bigg/ \frac{\delta F}{\delta Z_k}[0, 1] \tag{22}$$

where

$$F[g, h] = \int h^X \cdot G_k[g|X] \cdot f_{k|k-1}(X|Z^{(k-1)})\delta X \tag{23}$$

$$G_k[g|X] = \int g^Z \cdot f_k(Z|X)\delta Z \tag{24}$$

and where $G_k[g|X]$ is the PGFL of the multi-target likelihood $f_k(Z|X)$, $Z^{(k)} = \{Z_1, ..., Z_k\}$. If $X = \emptyset$, $h^X = 1$; otherwise, $h^X = \prod_{x \in X} h(x)$. For a single extended target with state x, the PGFL for the Poisson measurement model due to Gilholm et al. (Eq. (12.217), pp. 431 of [2]) is

$$G_k[g|x] = e^{\gamma(x) \cdot \phi_g(x) - \gamma(x)} \tag{25}$$

where

$$\phi_g(x) = \int g(z) \cdot \phi_z(x)\, dz \tag{26}$$

If the detection probability is taken into account, Eq. (25) becomes

$$G_k[g|x] = 1 - p_{D,k}(x) + p_{D,k}(x)\, e^{\gamma(x) \cdot \phi_g(x) - \gamma(x)} \tag{27}$$

Therefore, taking the Poisson clutter into account, Eq. (23) becomes

$$F[g,h] = e^{\lambda c[g]-\lambda} G_{k|k-1}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right] \tag{28}$$

where

$$c[g] = \int g(z)\, c_k(z) dz \tag{29}$$

In order to obtain the measurement-update equations of a CBMeMBer filter based on the Poisson extended-target measurement model, assume that the PGFL of the predicted density is a multi-Bernoulli

$$G_{k|k-1}[h] = \prod_{i=1}^{M_{k|k-1}} \left(1 - r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}[h]\right) \tag{30}$$

Then we get

$$\begin{aligned} F[g,h] = e^{\lambda c[g]-\lambda} \prod_{i=1}^{M_{k|k-1}} \Big(1 - r_{k|k-1}^{(i)} + \\ r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]\Big) \end{aligned} \tag{31}$$

By Eq. (22), we firstly derive the derivatives of $F[g,h]$ in Section IV.B. Then on the basis of Lemma 1, $G_k[h]$ is computed in Section IV.C. Finally, the posterior density π_k is approximated by a multi-Bernoulli in Section IV.D.

B. Derivatives of $F[g,h]$

LEMMA 1. *The derivatives of $F[g,h]$ is*

$$\frac{\delta F}{\delta Z_k}[g,h] \approx F[g,h] \cdot \Pi_{Z_k} \cdot \sum_{\wp \angle Z_k} \prod_{W\in\wp} d_W[g,h] \tag{32}$$

where

$$\Pi_{Z_k} = \prod\nolimits_{z\in Z_k} \lambda c(z) \tag{33}$$

$$\begin{aligned} & d_W[g,h] \approx \delta_{1,|W|} + \\ & \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \prod_{z\in W} \ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]} \end{aligned} \tag{34}$$

$$\ell_z(x) = \frac{\gamma(x)\,\phi_z(x)}{\lambda c(z)} \tag{35}$$

Lemma 1 is proved by induction on the number of elements in Z_k. First, consider the case of $Z_k = \{z_1\}$ with $|Z_k| = 1$, in which there is only one partition, i.e., $\wp = \{\{z_1\}\}$. Thus

$$\begin{aligned} & \tfrac{\delta F}{\delta z_1}[g,h] = F[g,h] \cdot (\lambda c(z_1) + \\ & \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \gamma\phi_{z_1}\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]}\Bigg) \\ & = F[g,h] \cdot \lambda c(z_1) \cdot (1 + \\ & \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \ell_{z_1}\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]}\Bigg) \end{aligned} \tag{36}$$

It can be seen that Lemma 1 is valid for $|Z_k| = 1$. Now, assume that Lemma 1 is valid for $Z_k = \{z_1, ..., z_m\}$ with $|Z| = m > 1$. We are to prove that Lemma 1 is still valid for $Z_k = \{z_1, ..., z_m, z_{m+1}\}$ with $|Z_k| = m+1$. The functional derivatives can be calculated by

$$\begin{aligned} \tfrac{\delta^2 F}{\delta Z_k \delta z_{m+1}}[g,h] = \tfrac{\delta F}{\delta z_{m+1}}[g,h] \cdot \Pi_{Z_k} \cdot \sum_{\wp\angle Z_k} \prod_{W\in\wp} d_W[g,h] \\ + F[g,h] \cdot \Pi_{Z_k} \cdot \sum_{\wp\angle Z_k} \tfrac{\delta}{\delta z_{m+1}} \prod_{W\in\wp} d_W[g,h] \end{aligned} \tag{37}$$

On the one hand,

$$\begin{aligned} & \tfrac{\delta F}{\delta z_{m+1}}[g,h] = F[g,h] \cdot \lambda c(z_{m+1}) \cdot (1 + \\ & \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \ell_{z_{m+1}}\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]}\Bigg) \end{aligned} \tag{38}$$

On the other hand,

$$\begin{aligned} \tfrac{\delta}{\delta z_{m+1}} \prod_{W\in\wp} d_W[g,h] = \left(\prod_{W\in\wp} d_W[g,h]\right) \cdot \\ \sum_{W\in\wp} \tfrac{1}{d_W[g,h]} \tfrac{\delta d_W}{\delta z_{m+1}}[g,h] \end{aligned} \tag{39}$$

and by Eq. (34),

$$\begin{aligned} & \tfrac{\delta d_W}{\delta z_{m+1}}[g,h] = \\ & \sum_{i=1}^{M_{k|k-1}} \Bigg(\frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \gamma\phi_{z_{m+1}} \prod_{z\in W} \ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]} + \\ & \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \prod_{z\in W} \ell_z\right]}{\left(1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]\right)^2} \cdot \\ & \qquad r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \gamma\phi_{z_{m+1}}\right]\Bigg) \end{aligned} \tag{40}$$

In general, we assume that the second term of Eq. (40) is small enough to ignore in the sense of a low clutter density. Thus Eq. (40) can be approximated by

$$\begin{aligned} & \tfrac{\delta d_W}{\delta z_{m+1}}[g,h] \approx \lambda c(z_{m+1}) \cdot \\ & \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h p_{D,k} e^{\gamma\phi_g-\gamma} \prod_{z\in W\cup\{z_{m+1}\}} \ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[h\left(1-p_{D,k}+p_{D,k}e^{\gamma\phi_g-\gamma}\right)\right]} \end{aligned} \tag{41}$$

Hence, Eq. (37) becomes

$$\begin{aligned} & \tfrac{\delta^2 F}{\delta Z_k \delta z_{m+1}}[g,h] \approx \\ & F[g,h] \cdot \Pi_{Z_k\cup\{z_{m+1}\}} \cdot \sum_{\wp\angle Z_k} \left(\prod_{W\in\wp\cup\{\{z_{m+1}\}\}} d_W[g,h] \right) + \\ & F[g,h] \cdot \Pi_{Z_k\cup\{z_{m+1}\}} \cdot \sum_{\wp\angle Z_k} \Bigg(\prod_{W\in\wp} d_W[g,h] \cdot \\ & \qquad \sum_{W\in\wp} \frac{d_{W\cup\{z_{m+1}\}}[g,h]}{d_W[g,h]} \Bigg) \end{aligned} \tag{42}$$

Note that all partitions of $Z_k \cup \{z_{m+1}\}$ can be obtained as follows. On the one hand, add the cell $\{z_{m+1}\}$ to a partition $\wp$ of Z_k to obtain a new partition, i.e., $\wp_1 = \wp \cup \{\{z_{m+1}\}\}$. This case corresponds to the first term of Eq. (42). On the other hand, remove a cell W from $\wp$ and replace it with $W \cup \{z_{m+1}\}$ to obtain a new partition, i.e., $\wp_2 = \{W \cup \{z_{m+1}\}\} \cup \{\cup_{W'\in\wp-\{W\}} \{W'\}\}$. This case corresponds to the second term of Eq. (42). Thus Eq. (42) can be rewritten as

$$\frac{\delta^2 F}{\delta Z_k \delta z_{m+1}}[g,h] \approx F[g,h]\cdot \Pi_{Z_k\cup\{z_{m+1}\}} \cdot \sum_{\wp\angle Z_k\cup\{z_{m+1}\}} \left(\prod_{W\in\wp} d_W[g,h]\right) \tag{43}$$

It can be seen that Lemma 1 is still valid for $|Z_k| = m+1$. The inductive process is over.

C. Formula for $G_k[h]$

LEMMA 2. *On the basis of Lemma 1, the PGFL of the multiple extended-target posterior density is*

$$G_k[h] = G_{L,k}[h]\cdot G_{U,k}[h] \tag{44}$$

where

$$G_{L,k}[h] = \prod_{i=1}^{M_{k|k-1}} \frac{1-r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}[h(1-\bar{p}_{D,k})]}{1-r_{k|k-1}^{(i)} + r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}[1-\bar{p}_{D,k}]} \tag{45}$$

$$G_{U,k}[h] = \sum\nolimits_{\wp\angle Z_k} G_{U,k}[h;\wp] \tag{46}$$

$$G_{U,k}[h;\wp] = \frac{\prod_{W\in\wp} d_W[h]}{\sum_{\wp'\angle Z_k}\prod_{W\in\wp'} d_W} \tag{47}$$

$$d_W[h] = \delta_{1,|W|} + \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[hp_{D,k}e^{-\gamma}\prod_{z\in W}\ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)}p_{k|k-1}^{(i)}[h(1-\bar{p}_{D,k})]} \tag{48}$$

$$d_W = d_W[1] = \delta_{1,|W|} + \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[p_{D,k}e^{-\gamma}\prod_{z\in W}\ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)}p_{k|k-1}^{(i)}[1-\bar{p}_{D,k}]} \tag{49}$$

Lemma 2 is proved as follows. By Eq. (22) and Lemma 1, we get

$$\begin{aligned} G_k[h] &= \frac{\delta F}{\delta Z_k}[0,h] \Big/ \frac{\delta F}{\delta Z_k}[0,1] \\ &= \frac{F[0,h]}{F[0,1]}\cdot \frac{\sum_{\wp\angle Z_k}\prod_{W\in\wp} d_W[0,h]}{\sum_{\wp'\angle Z_k}\prod_{W\in\wp'} d_W[0,1]} \\ &= \prod_{i=1}^{M_{k|k-1}} \frac{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)}p_{k|k-1}^{(i)}[h(1-\bar{p}_{D,k})]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)}p_{k|k-1}^{(i)}[1-\bar{p}_{D,k}]} \\ &\quad\cdot \frac{\sum_{\wp\angle Z_k}\prod_{W\in\wp} d_W[0,h]}{\sum_{\wp'\angle Z_k}\prod_{W\in\wp'} d_W[0,1]} \end{aligned} \tag{50}$$

and

$$d_W[0,h] = d_W[h] = \delta_{1,|W|} + \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}\left[hp_{D,k}e^{-\gamma}\prod_{z\in W}\ell_z\right]}{1-r_{k|k-1}^{(i)}+r_{k|k-1}^{(i)}p_{k|k-1}^{(i)}[h(1-\bar{p}_{D,k})]} \tag{51}$$

Thus the proof of Lemma 2 is completed.

D. Proof of Theorem 1

In terms of $G_k[h]$, we hope it is still a multi-Bernoulli. $G_{L,k}[h]$ denotes the legacy tracks, and is a multi-Bernoulli

$$G_{L,k}[h] = \prod_{i=1}^{M_{k|k-1}} \left(1-r_{L,k}^{(i)} + r_{L,k}^{(i)} p_{L,k}^{(i)}[h]\right) \tag{52}$$

where $r_{L,k}^{(i)}$ and $p_{L,k}^{(i)}$ can be seen in Eq. (12) and Eq. (13), respectively.

$G_{U,k}[h]$ denotes the measurement-corrected tracks, but is not a multi-Bernoulli. Thus, we propose to approximate the $G_{U,k}[h]$ by a multi-Bernoulli as in [15].

First, substituting $y = h(x)$ into $G_{U,k}[h;\wp]$, and differentiating it at $y=1$, we can exactly obtain the mean cardinality,

$$\sum\nolimits_{\wp\angle Z} G'_{U,k}[h;\wp] = \sum\nolimits_{\wp\angle Z}\sum\nolimits_{W\in\wp} r_{U,k}(W) \tag{53}$$

where $r_{U,k}(W)$ can be seen in Eq. (14).

$\upsilon_{U,k}(x;\wp)$ denotes the intensity of $G_{U,k}[h;\wp]$. $G_{U,k}[h;W]$ denotes the PGFL for a cell W and $\upsilon_{U,k}(x;W)$ is its intensity. Then we will seek a Bernoulli

$$1 - r_{U,k}(W) + r_{U,k}(W)\, p_{U,k}(\cdot;W)[h]$$

to approximate $G_{U,k}[h;W]$, and ensure that they have the same intensity function, i.e.,

$$r_{U,k}(W)\, p_{U,k}(\cdot;W) = \upsilon_{U,k}(\cdot;W) \tag{54}$$

The intensity function $\upsilon_{U,k}(x;\wp)$ can be obtained by calculating the functional derivative of $G_{U,k}[h;\wp]$ at x (Eq. (11.186), pp. 375 of [2]),

$$\upsilon_{U,k}(x;\wp) = \sum\nolimits_{W\in\wp} \upsilon_{U,k}(x;W) \tag{55}$$

where

$$\upsilon_{U,k}(x;W) = \frac{w_\wp}{d_W}\sum\nolimits_{i=1}^{M_{k|k-1}} \upsilon_{U,k}^{(i)}(x;W) \tag{56}$$

and where $\upsilon_{U,k}^{(i)}(x;W)$ can be seen in Eq. (20).

Finally, by Eq. (54), we can get $p_{U,k}(x;W)$ (see Eq. (15)).

V. CONCLUSION

The CBMeMBer filter for extended targets is studied in this paper. We established the measurement-update equations for a CBMeMBer recursion based on the Poisson extended-target measurement model due to Gilholm et al. In the future work, we will focus on the tractable approach.

ACKNOWLEDGMENT

This work was supported by Foundation for Innovative Research Groups of the National Natural Science Foundation of China (61221063), Grant for State Key Program for Basic Research of China (973) (No. 2013CB329405).

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